INVESTIGATION 3-A

Orbital Speed of Planets

TARGET SKILLS
- Modelling concepts
- Analyzing and interpreting
- Communicating results

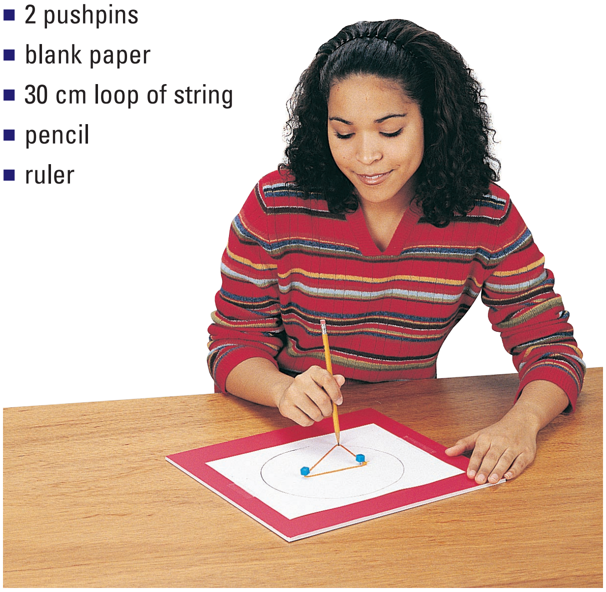
Can you show diagrammatically that a force directed along the line between the centres of the Sun and a planet would cause the planet's speed to increase as it approached the Sun and decrease as it moved away? If you can, you have demonstrated that Kepler's second law supports Newton's proposed law of universal gravitation.

Problem

How does a force that follows an inverse square relationship affect the orbital speed of a planet in an elliptical orbit?

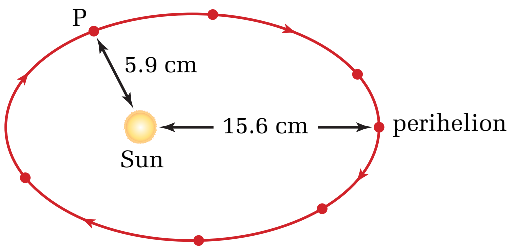
Equipment

- corkboard or large, thick piece of cardboard
- 2 pushpins
- blank paper
- 30 cm loop of string
- pencil
- ruler



Procedure

1. Place the paper on the corkboard or cardboard. Insert two pushpins into the paper about 8 to 10 cm apart.
2. Loop the string around the pushpins, as shown in the illustration. With your pencil, pull the string so that it is taut and draw an ellipse by pulling the string all the way around the pushpins.
3. Remove the string and pushpins and label one of the pinholes "Sun."
4. Choose a direction around the elliptical orbit in which your planet will be moving. Make about four small arrowheads on the ellipse to indicate the direction of motion of the planet.
5. Make a dot for the planet at the point that is most distant from the Sun (the perihelion). Measure and record the distance on the paper from the perihelion to the Sun. From that point, draw a 1 cm vector directed straight toward the Sun.
6. This vector represents the force of gravity on the planet at that point: $F_{g(per)} = 1$ unit. ($F_{g(per)}$ is the force of gravity when the planet is at perihelion.)
7. Select and label at least three more points on each side of the ellipse at which you will analyze the force and motion of the planet.
8. For each point, measure and record, on a separate piece of paper, the distance from the Sun to point P, as indicated in the diagram. Do not write on your diagram, because it will become too cluttered.



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9. Follow the steps in the table to see how to determine the length of the force vector at each point.

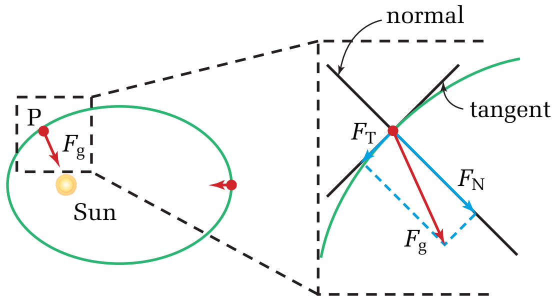
Procedure	Equation
■ The masses of the Sun and planet remain the same, so the value Gm_Sm_p is constant. Therefore, the expression F_gr^2 for any point on the orbit is equal to the same value.	$F_g = G\frac{m_Sm_p}{r^2}$ $F_gr^2 = Gm_Sm_p$
■ Consequently, you can set the expression F_gr^2 for any one point equal to F_gr^2 for any other point. Use the values at perihelion as a reference and set $F_{g(P)}r^2$ equal to $F_{g(peri)}r^2_{peri}$. Then solve for the $F_{g(P)}$.	$F_{g(peri)}\, r^2_{peri} = F_{g(P)}\, r^2_P$ $F_{g(P)} = \frac{F_{g(peri)}\, r^2_{peri}}{r^2_P}$
■ You can now find the relative magnitude of the gravitational force on the planet at any point on the orbit by substituting the magnitudes of the radii into the above equation. For example, the magnitude of the force at point P in step 8 is 6.99 units.	$F_{g(P)} = \frac{(1\text{ unit})(15.6\text{ cm})^2}{(5.9\text{ cm})^2}$ $F_{g(P)} = 6.99$ units

10. Calculate the length of the force vector from each of the points that you have selected on your orbit.

11. On your diagram, draw force vectors from each point directly toward the Sun, making the lengths of the vectors equal to the values that you calculated in step 10.

12. At each point at which you have a force vector, draw a very light pencil line tangent to the ellipse. Then, draw a line that is perpendicular (normal) to the tangent line.

13. Graphically draw components of the force vector along the tangent (F_T) and normal (F_N) lines, as shown in the diagram.



Analyze and Conclude

1. The tangential component of the force vector (F_T) is parallel to the direction of the velocity of the planet when it passes point P. What effect will the tangential component of force have on the velocity of the planet?

2. The normal component of the force vector (F_N) is perpendicular to the direction of the velocity of the planet when it passes point P. What effect will the normal component of force have on the velocity of the planet?

3. Analyze the change in the motion of the planet caused by the tangential and normal components of the gravitational force at each point where you have drawn force vectors. Be sure to note the direction of the velocity of the planet as you analyze the effect of the components of force at each point.

4. Summarize the changes in the velocity of the planet as it makes one complete orbit around the Sun.

5. The force vectors and components that you drew were predictions based on Newton's law of universal gravitation. How well do these predictions agree with Kepler's observations as summarized in his second law? Would you say that Kepler's data supports Newton's predictions?

Kepler's third law simply states that the ratio T^2/r^3 is constant and the same for each planet orbiting the Sun. At first glance, it would appear to have little relationship to Newton's law of universal gravitation, but a mathematical analysis will yield a relationship. To keep the mathematics simple, you will consider only circular orbits. The final result obtained by considering elliptical orbits is the same, although the math is more complex. Follow the steps below to see how Newton's law of universal gravitation yields the same ratio as given by Kepler's third law.

- Write Newton's law of universal gravitation, using m_S for the mass of the Sun and m_p for the mass of a planet.

$$F_g = G\frac{m_S m_p}{r^2}$$

- Since the force of gravity must provide a centripetal force for the planets, set the gravitational force equal to the required centripetal force.

$$G\frac{m_S m_p}{r^2} = \frac{m_p v^2}{r}$$

  Simplify the equation.

$$G\frac{m_S}{r} = v^2$$

- Since Kepler's third law includes the period, T, as a variable, find an expression for the velocity, v, of the planet in terms of its period.

$$v = \frac{\Delta d}{\Delta t}$$

$$\Delta d = 2\pi r$$

$$\Delta t = T$$

  A planet travels a distance equal to the circumference of the orbit during a time interval equal to its period.

$$v = \frac{2\pi r}{T}$$

- Substitute the expression for the velocity of the planet into the above equation.

$$G\frac{m_S}{r} = \left(\frac{2\pi r}{T}\right)^2$$

$$G\frac{m_S}{r} = \frac{4\pi^2 r^2}{T^2}$$

- Multiply each side of the equation by T^2/r^2.

$$\left(G\frac{m_S}{r}\right)\left(\frac{T^2}{r^2}\right) = \left(\frac{4\pi^2 \cancel{r^2}}{\cancel{T^2}}\right)\left(\frac{\cancel{T^2}}{\cancel{r^2}}\right)$$

$$\frac{G m_S T^2}{r^3} = 4\pi^2$$

- Solve for T^2/r^3.

$$\frac{T^2}{r^3} = \frac{4\pi^2}{G m_S}$$

As you can see, Newton's law of universal gravitation indicates not only that the ratio T^2/r^3 is constant, but also that the constant is $4\pi^2/Gm_S$. All of Kepler's laws, developed prior to the time when

Newton did his work, support Newton's law of universal gravitation. Kepler had focussed only on the Sun and planets, but Newton proposed that the laws applied to all types of orbital motion, such as moons around planets. Today, we know that all of the artificial satellites orbiting Earth, as well as the Moon, follow Kepler's laws.

Mass of the Sun and Planets

Have you ever looked at tables that contain data for the mass of the Sun and planets and wondered how anyone could "weigh" the Sun and planets or determine their masses? English physicist and chemist Henry Cavendish (1731–1810) realized that if he could determine the universal gravitational constant, G, he could use the mathematical relationship in Kepler's third law to calculate the mass of the Sun. A brilliant experimentalist, Cavendish designed a torsion balance, similar to the system in Figure 3.4, that allowed him to measure G.

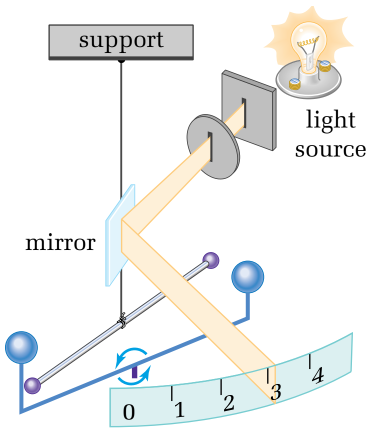


Figure 3.4 In the torsion balance that Cavendish designed and used, the spheres were made of lead. The small spheres were about 5 cm in diameter and were attached by a thin but rigid rod about 1.83 m long. The large spheres were about 20 cm in diameter.

A torsion balance can measure extremely small amounts of the rotation of a wire. First, the torsion balance must be calibrated to determine the amount of force that causes the wire to twist by a specific amount. Then, the large spheres are positioned so that the bar supporting them is perpendicular to the rod supporting the small spheres. In this position, the large spheres are exerting equal gravitational attractive forces on each of the small spheres. The system is in equilibrium and the scale can be set to zero. The large spheres are then moved close to the small spheres and the amount of twisting of the wire is determined. From the amount of twisting and the calibration, the mutual attractive force between the large and small spheres is calculated.

Using his torsion balance, Cavendish calculated the value of G to be $6.75 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$. The best-known figure today is $6.672\ 59 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$. Cavendish's measurement was within approximately 1% of the correct value. As Cavendish did, you can now calculate the mass of the Sun and other celestial bodies.

HISTORY LINK

Henry Cavendish was a very wealthy and brilliant man, but he also was very reclusive. He was rarely seen in public places, other than at scientific meetings. His work was meticulous, yet he published only a very small part of it. After his death, other scientists discovered his notebooks and finally published his results. Cavendish had performed the same experiments and obtained the same results for some experiments that were later done by Coulomb, Faraday, and Ohm, who received the credit for the work.

PHYSICS FILE

The value of G shows that the force of gravity is extremely small. For example, use unit amounts of each of the variables and substitute them into Newton's law of universal gravitation. You will find that the mutual attractive force between two 1 kg masses that are 1 m apart is $6.672\ 59 \times 10^{-11}$ N.

SAMPLE PROBLEM

The Mass of the Sun

Find the mass of the Sun, using Earth's orbital radius and period of revolution.

Conceptualize the Problem

- Kepler's third law, combined with Newton's law of universal gravitation, yields an equation that relates the period and orbital radius of a satellite to the mass of the body around which the satellite is orbiting.
- Earth orbits the Sun once per year.
- Let R_E represent the radius of Earth's orbit around the Sun. This value can be found in Appendix B, Physical Constants and Data.

Identify the Goal

The mass of the Sun, m_S

Identify the Variables and Constants

Known	Implied	Unknown
Sun	$G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$	m_S
	$T = 365.25$ days	
	$R_{E(orbit)} = 1.49 \times 10^{11}$ m	

Develop a Strategy

Write Kepler's third law, using the constant derived from Newton's law of universal gravitation.

$$\frac{T^2}{r^3} = \frac{4\pi^2}{Gm_S}$$

Solve for the mass of the Sun.

$$\frac{T^2}{r^3} m_S = \frac{4\pi^2}{G\cancel{m_S}} \cancel{m_S}$$

$$m_S = \left(\frac{4\pi^2}{G}\right)\left(\frac{r^3}{T^2}\right)$$

Convert the period into SI units.

$$365.25\ \cancel{\text{days}} \left(\frac{24\ \cancel{h}}{\cancel{\text{day}}}\right)\left(\frac{60\ \cancel{\text{min}}}{\cancel{h}}\right)\left(\frac{60\ s}{\cancel{\text{min}}}\right) = 3.1558 \times 10^7\ s$$

Substitute the numerical values into the equation and solve.

$$m_S = \left(\frac{4\pi^2}{6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2}}\right)\frac{(1.49 \times 10^{11}\ m)^3}{(3.1558 \times 10^7\ s)^2}$$

$$m_S = 1.9660 \times 10^{30}\ kg$$

$$m_S \cong 1.97 \times 10^{30}\ kg$$

The mass of the Sun is approximately 1.97×10^{30} kg.

Validate the Solution

The Sun is much more massive than any of the planets. The value sounds reasonable.

Check the units: $\left(\frac{1}{\frac{N \cdot m^2}{kg^2}}\right)\left(\frac{m^3}{s^2}\right) = \left(\frac{kg^2}{N \cdot \cancel{m^2}}\right)\left(\frac{\cancel{m^3}}{s^2}\right) = \left(\frac{kg^{\cancel{2}}}{\frac{\cancel{kg} \cdot \cancel{m}}{\cancel{s^2}} \cdot \cancel{m^2}}\right)\left(\frac{\cancel{m^3}}{\cancel{s^2}}\right) = kg$.

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PRACTICE PROBLEMS

9. Jupiter's moon Io orbits Jupiter once every 1.769 days. Its average orbital radius is 4.216×10^8 m. What is Jupiter's mass?

10. Charon, the only known moon of the planet Pluto, has an orbital period of 6.387 days at an average distance of 1.9640×10^7 m from Pluto. Use Newton's form of Kepler's third law to find the mass of Pluto from this data.

11. Some weather satellites orbit Earth every 90.0 min. How far above Earth's surface is their orbit? (Hint: Remember that the centre of the orbit is the centre of Earth.)

12. How fast is the moon moving as it orbits Earth at a distance of 3.84×10^5 km?

13. On each of the *Apollo* lunar missions, the command module was placed in a very low, approximately circular orbit above the Moon. Assume that the average height was 60.0 km above the surface and that the Moon's radius is 7738 km.
    (a) What was the command module's orbital period?
    (b) How fast was the command module moving in its orbit?

14. A star at the edge of the Andromeda galaxy appears to be orbiting the centre of that galaxy at a speed of about 2.0×10^2 km/s. The star is about 5×10^9 AU from the centre of the galaxy. Calculate a rough estimate of the mass of the Andromeda galaxy. Earth's orbital radius (1 AU) is 1.49×10^8 km.

Newton's law of universal gravitation has stood the test of time and the extended limits of space. As far into space as astronomers can observe, celestial bodies move according to Newton's law. As well, the astronauts of the crippled *Apollo 13* spacecraft owe their lives to the dependability and predictability of the Moon's gravity. Although Albert Einstein (1879–1955) took a different approach in describing gravity in his general theory of relativity, most calculations that need to be made can use Newton's law of universal gravitation and make accurate predictions.

3.1 Section Review

1. K/U Explain the meaning of the term "empirical" as it applies to empirical equations.

2. K/U What did Tycho Brahe contribute to the development of the law of universal gravitation?

3. K/U Describe how Newton used each of the following phenomena to support the law of universal gravitation.
    (a) the orbit of the moon
    (b) Kepler's third law

4. K/U How did Newton's concepts about gravity and his development of the law of universal gravitation differ from the ideas of other scientists and astronomers who were attempting to find a relationship that could explain the motion of the planets?

5. K/U Describe the objective, apparatus, and results of the Cavendish experiment.

6. C Explain how you can "weigh" a planet.

7. I Suppose the distance between two objects is doubled and the mass of one is tripled. What effect does this have on the gravitational force between the objects?